13. marzo. 2018

RECTAS, PLANOS Y CURVAS EN EL ESPACIO

RECTAS

1) $y = mx + b$ Pendiente ordenada al origen.
2) $y - y_1 = m(x - x_1)$ Punto pendiente.
3) $Ax + By + C = 0$ Forma general.

$\in \mathbb{R}^2$

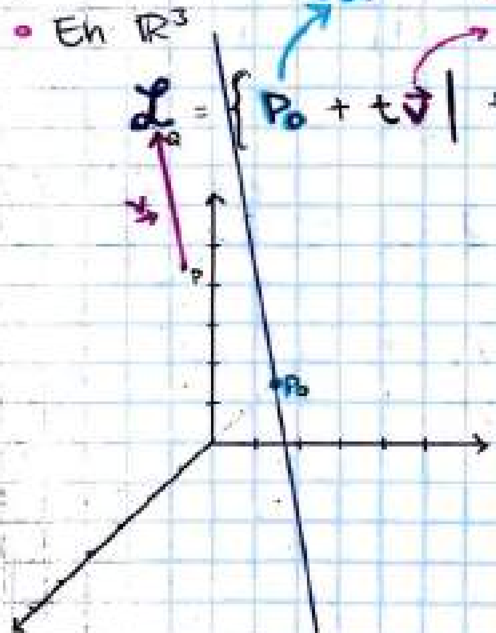
- En $\mathbb{R}^3$

Punto $\in \mathcal{L}$

Vector director

FORMA VECTORIAL

$$\mathcal{L} = \{ P_0 + t\vec{v} \mid t \in \mathbb{R},\ P_0 \in \mathbb{R}^3,\ \vec{v} \in V^3 \}$$



$P = (-2, -2, 3)$
$Q = (-4, -4, 5)$

$$P_0 = (\overset{x_0}{2}, \overset{y_0}{3}, \overset{z_0}{3})$$

$$\vec{v} = \langle \underset{a}{-2}, \underset{b}{-2}, \underset{c}{2} \rangle = \overline{PQ}$$

FORMA PARAMÉTRICA

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

$x = 2 - 2t \rightarrow t = \frac{x-2}{-2}$
$y = 3 - 2t \rightarrow t = \frac{y-3}{-2}$
$z = 3 + 2t \rightarrow t = \frac{z-3}{2}$

FORMA SIMÉTRICA

$$\frac{x-2}{-2} = \frac{y-3}{-2} = \frac{z-3}{2}$$

Existe siempre y cuando $a, b, c \neq 0$.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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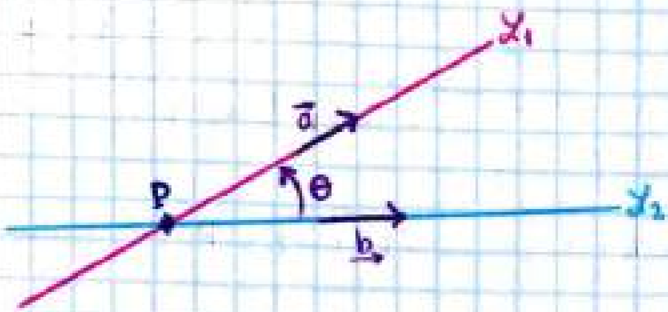
→ Para cada par de puntos distintos de $\mathbb{R}^3$ hay una y sólo una recta que pasa por ellos.

Def. 2 rectas

$$\mathcal{L}_1 = \{ P_1 + s\vec{a} \mid s \in \mathbb{R} \} \quad y$$
$$\mathcal{L}_2 = \{ P_2 + t\vec{b} \mid t \in \mathbb{R} \}$$

se dicen paralelas si los vectores $\vec{a}$ y $\vec{b}$ son paralelos.



① $\mathcal{L}_1 \parallel \mathcal{L}_2$

② $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$

③ $\mathcal{L}_1 \cap \mathcal{L}_2 = P$

$$\boxed{\theta = \arccos \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}}$$

• Si las rectas $\mathcal{L}_1$ y $\mathcal{L}_2$ <u>NO</u> son paralelas, entonces la $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$ o bien $\mathcal{L}_1 \cap \mathcal{L}_2 = P$

EJEMPLO 1 - Determine si las rectas $\mathcal{L}_1$ y $\mathcal{L}_2$ son paralelas o no y diga cuál es su intersección

$$\mathcal{L}_1 = \{ (1,3,-2) + t\underbrace{\langle 3,-6,9 \rangle}_{\vec{a}} \}$$
$$\mathcal{L}_2 = \{ (2,1,7) + s\underbrace{\langle -2,4,-6 \rangle}_{\vec{b}} \}$$

¿$\vec{a} \parallel \vec{b}$? $\quad \vec{a} = \alpha\vec{b}$

$$\langle 3,-6,9 \rangle = \alpha \langle -2,4,-6 \rangle$$

$$\left.\begin{array}{l} -2\alpha = 3 \\ 4\alpha = -6 \\ -6\alpha = 9 \end{array}\right\} \alpha = -3/2$$

$$\boxed{\therefore \vec{a} = -3/2\,\vec{b} \quad \text{SON PARALELAS}}$$

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2) Determine si $\mathcal{L}_1$ y $\mathcal{L}_2$ son o no son paralelas y determine su intersección (si existe).

$$\mathcal{L}_1 = \{(1,3,-2) + t\langle 3, \underset{\vec{a}}{-6}, 9\rangle\}$$

$$\mathcal{L}_2 = \{(2,1,7) + s\langle 1, \underset{\vec{b}}{-3}, 4\rangle\}$$

a) $\vec{a} = \alpha\vec{b} \implies \langle 3,-6,9\rangle = \alpha\langle 1,-3,4\rangle$

$$\left.\begin{array}{l} \alpha = 3 \\ -3\alpha = -6 \longrightarrow \alpha = 2 \\ 4\alpha = 9 \longrightarrow \alpha = 9/4 \end{array}\right\}$$ No existe α que cumpla $\vec{a} = \alpha\vec{b}$

$$\boxed{\therefore \mathcal{L}_1 \not\parallel \mathcal{L}_2}$$

b) $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$ o bien $\mathcal{L}_1 \cap \mathcal{L}_2 = P$

$$P_0 = (1,3,-2) + t\langle 3,-6,9\rangle = (2,1,7) + s\langle 1,-3,4\rangle$$

$$t\langle 3,-6,9\rangle - s\langle 1,-3,4\rangle = (2,1,7) - (1,3,-2)$$

$$\begin{array}{l} 3t - s = 1 \\ -6t + 3s = -2 \\ 9t - 4s = 9 \end{array}$$

$$\begin{array}{r} 9t - 3s = 3 \\ -6t + 3s = -2 \\ \hline 3t = 1 \\ \boxed{t = 1/3} \end{array}$$

$$3\left(\tfrac{1}{3}\right) - s = 1$$
$$1 - 1 = s$$
$$\boxed{s = 0}$$

$$9\left(\tfrac{1}{3}\right) - 4(0) = 9$$
$$3 - 0 = 9$$
$$\boxed{3 \neq 9}$$ → No hay intersección entre las rectas

$$\boxed{\therefore \mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset}$$

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(3) Hallar el $\measuredangle$ entre las rectas $\mathcal{L}_1$ y $\mathcal{L}_2$ del ejemplo anterior

$$\left.\begin{aligned} \mathcal{L}_1 &= \vec{a} = \langle 3, -6, 9 \rangle \\ \mathcal{L}_2 &= \vec{b} = \langle 1, -3, 4 \rangle \end{aligned}\right\} \text{Vectores directores.}$$

$$\vec{a} \cdot \vec{b} = 3 + 18 + 36 = 57$$

$$\|\vec{a}\| = \sqrt{3^2 + (-6)^2 + 9^2} = \sqrt{9 + 16 + 81} = \sqrt{126}$$

$$\|\vec{b}\| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{1 + 9 + 16} = \sqrt{26}$$

$$\theta = \arccos\left(\frac{57}{\sqrt{126}\sqrt{26}}\right) = 0.99587 \text{ rad}$$

$$\boxed{\theta = 5.20^\circ}$$

- Si $\mathcal{L}_1 = \{P_1 + t\vec{a} \mid t \in \mathbb{R}\}$ y $\mathcal{L}_2 = \{P_2 + s\vec{b} \mid s \in \mathbb{R}\}$ son rectas paralelas, entonces $\mathcal{L}_1 \neq \mathcal{L}_2$, o bien, $\mathcal{L}_1 \cap \mathcal{L}_2 = \varnothing$

Planos

15. marzo. 2018

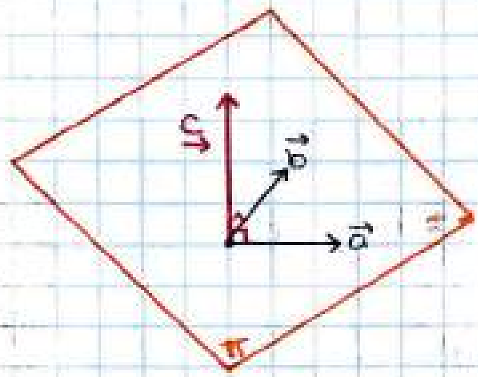
$$\boxed{\pi = \{ P_0 + t\vec{a} + s\vec{b} \mid t, s \in \mathbb{R} \}}$$

plano

Forma vectorial

$P_0 \in \pi$, $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$

$\vec{a}, \vec{b} \in V_3$ $\qquad \vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{a}, \vec{b} \in \pi$ $\qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$



$\vec{a} \times \vec{b} = \boxed{\vec{c}}$

$\vec{n}$ Vector normal.

Ecuaciones Paramétricas del Plano.

$$x = x_0 + a_1 t + b_1 s$$
$$y = y_0 + a_2 t + b_2 s$$
$$z = z_0 + a_3 t + b_3 s.$$

$$\boxed{\overrightarrow{P_0P} \cdot \underline{n} = 0}$$

Ecuación vectorial del plano

$P = (x, y, z)$
$P_0 = (x_0, y_0, z_0)$

$\overrightarrow{P_0P} = \langle x - x_0 , y - y_0 , z - z_0 \rangle$

$$\langle x - x_0 , y - y_0 , z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0.$$

$$d = -(ax_0 + by_0 + cz_0)$$

$$\boxed{ax + by + cz + d = 0}$$

Forma general del Plano

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Reglas Nemotécnicas

① $\begin{vmatrix} x-x_0 & a_1 & b_1 \\ y-y_0 & a_2 & b_2 \\ z-z_0 & a_3 & b_3 \end{vmatrix} = 0$

② $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$ → Teorema: Tres puntos no colineales determinan un plano único.

$\left.\begin{matrix} P_1 = (x_1, x_2, x_3) \\ P_2 = (y_1, y_2, y_3) \\ P_3 = (z_1, z_2, z_3) \end{matrix}\right\} \in \mathbb{R}^3$

EJEMPLO p. 802 Larson

- Hallar las diferentes formas del plano asociado a los puntos $(2,1,1)$, $(0,4,1)$ y $(-2,1,4)$.

$P_1 = (0,4,1)$ $\quad\vec{a} = \overline{P_1P_2} = (2,-3,0)$

$P_2 = (2,1,1)$ $\quad\vec{b} = \overline{P_1P_3} = (-2,-3,3)$

$P_3 = (-2,1,4)$

① $\pi = \{(0,4,1) + t(2,1,1) + s(-2,1,4)\}$

② $$\begin{aligned} x &= x_0 + a_1 t + b_1 s = 0 + 2t - 2s \\ y &= y_0 + a_2 t + b_2 s = 4 - 3t - 3s \\ z &= z_0 + a_3 t + b_3 s = 1 + 0t + 3s \end{aligned}$$

③ $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 0 \\ -2 & -3 & 3 \end{vmatrix} =$

$\hat{\imath}[-9] + \hat{\jmath}[6] + \hat{k}[-6-6] = \langle -9, 6, -12 \rangle = \vec{n}$

Tomamos a P_3 como P_0 $\quad \overline{P_3P} \cdot \vec{n} = 0$

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$$\overrightarrow{P_3P} = \left[(x, y, z) - (-2, 1, 4) \right]$$

$$P_3P = \langle x+2,\ y-1,\ z-4 \rangle$$

$$\overrightarrow{P_3P} \cdot \vec{n} = \langle x+2,\ y-1,\ z-4 \rangle \cdot \langle -9, -6, -12 \rangle = 0$$

$$= -9(x+2) - 6(y-1) - 12(z-4) = 0$$

$$= -9x - 6y - 12z \underbrace{- 18 + 6 + 48} = 0$$

$$= -9x - 6y - 12z + 36 = 0 \qquad (\div -3)$$

$$= 3x + 2y + 4z - 12 = 0$$

(4) $$\boxed{3x + 2y + 4z = 12}$$

(5) $$\begin{vmatrix} x-2 & 2 & -2 \\ y-1 & -3 & -3 \\ z-1 & 0 & 3 \end{vmatrix} = 0$$

$$(z-1)\left[-6-6\right] + 3\left[(x-2)(-3) - (2)(y-1)\right] = 0$$

$$(z-1)\left[-12\right] + 3\left[-3x+6-2y+2\right] = 0$$

$$\cancel{-12z} + 12 \cancel{-9x} + 18 \cancel{-6y} + 6 = 0$$

$$-9x - 6y - 12z + 36 = 0 \qquad (\div -3)$$

$$\boxed{3x + 2y + 4z = 12}$$

(6) $$\begin{vmatrix} x & y & z & 1 \\ 0 & 4 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -2 & 1 & 4 & 1 \end{vmatrix} = x \begin{vmatrix} 4 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix} + 2 \begin{vmatrix} y & z & 1 \\ 4 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix} + 2 \begin{vmatrix} y & z & 1 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= x\left\{4[1-4] \cancel{-1[1-1]} + 1[4-1]\right\} + 2\left\{y[1-4] - z[4-1] + 1[16-1]\right\}$$
$$+ 2\left\{\cancel{y[1-1]} - z[4-1] + 1[4-1]\right\} = 0$$

$$= x\{-12+3\} + 2\{-3y-3z+15\} + 2\{-3z+3\} = 0$$

$$= -9x - 6y - 12z + 36 \Rightarrow \boxed{3x + 2y + 4z = 12}$$

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EJERCICIO

1. Identifíquese cada uno de los siguientes planos.

a) $3(x-5) - 2(y+4) + 4(z-2) = 0$

b) $2x + 3y = 2$

c) $x - 2y + z = 0$

2. Determínese la recta que pasa el punto $(1,-5,6)$ paralelo a la normal al plano que contiene a los puntos $(0,1,2)$, $(3,2,6)$ y $(-2,0,5)$.

RESPUESTAS

1.

a) $3(x-5) - 2(y+4) + 4(z-2) = 0$

$\langle x-5,\ y+4,\ z-2 \rangle \cdot \langle 3,-2,4 \rangle = 0$

$P = (x,y,z)$

$P_0 = (5,-4,2)$ $\quad \vec{n} = \langle 3,-2,4 \rangle$

b) $2x + 3y = 2$

$2x + 3y - 2 = 0$

$(2x-1) + (3y-1) = 0$

$2(x-\frac{1}{2}) + 3(y-\frac{1}{3}) = 0$

$P = (x,y,z)$

$P_0 = (\frac{1}{2}, \frac{1}{3}, 0)$ $\quad \vec{n} = \langle 2,3,0 \rangle$

c) $x - 2y + z = 0$

$1(x-0) - 2(y-0) + 1(z-0) = 0$

$P = (x,y,z)$

$P_0 = (0,0,0)$ $\quad \vec{n} = \langle 1,-2,1 \rangle$

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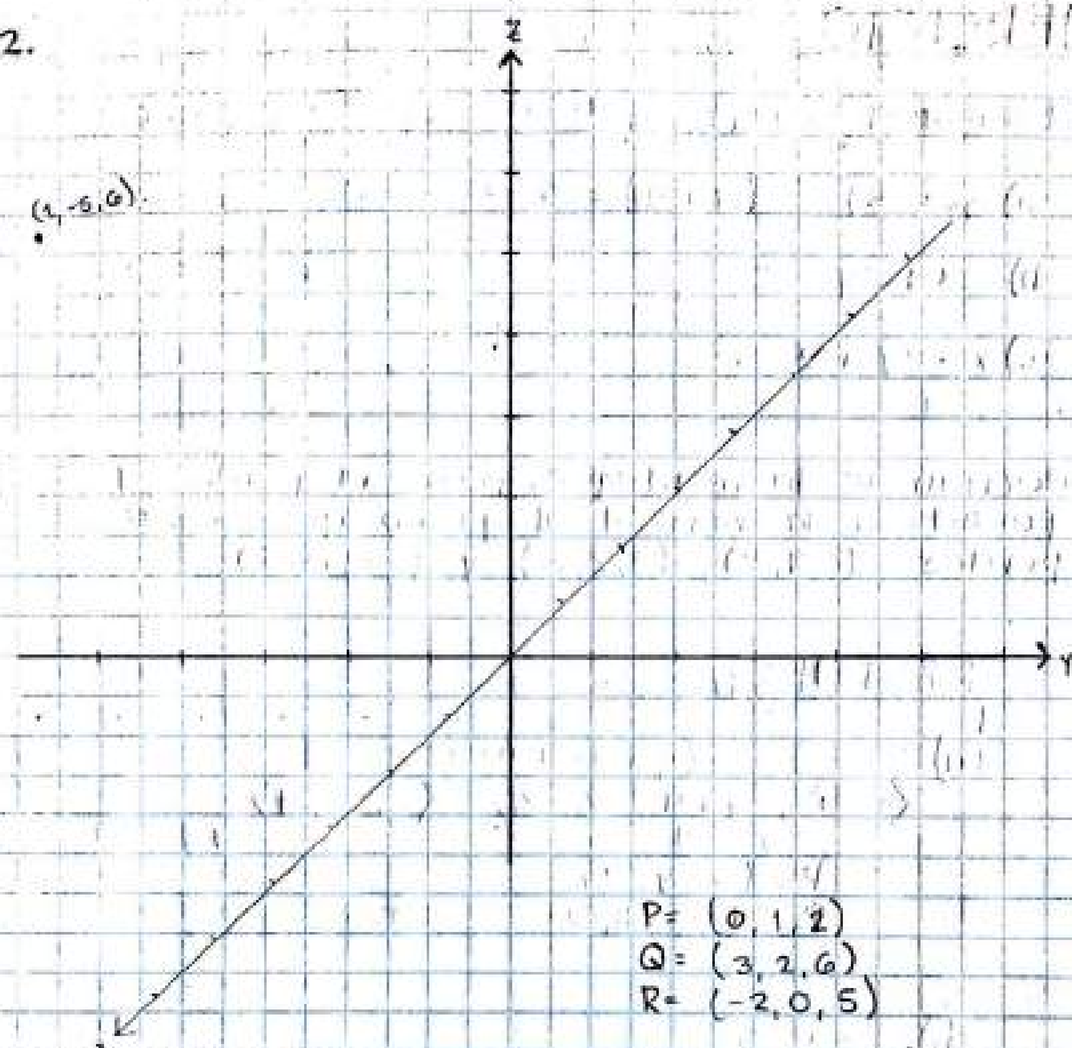
2.



$P = (0,1,2)$
$Q = (3,2,6)$
$R = (-2,0,5)$

$\overline{PQ} = (3,2,6) - (0,1,2)$
$\vec{a} = \langle 3,1,4 \rangle$

$\overline{PR} = (-2,0,5) - (0,1,2)$
$\vec{b} = \langle -2,-1,3 \rangle$

$$\pi = \{ (0,1,2) + t\langle 3,1,4\rangle + s\langle -2,1,3\rangle \}$$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 4 \\ -2 & -1 & 3 \end{vmatrix} = \hat{\imath}(3+4) - \hat{\jmath}(9+8) + \hat{k}(-3+2)$$
$$= 3\hat{\imath} + 4\hat{\imath} - 9\hat{\jmath} - 8\hat{\jmath} - 3\hat{k} + 2\hat{k}$$
$$= 7\hat{\imath} - 17\hat{\jmath} - \hat{k} = \boxed{\langle 7,-17,1\rangle = \vec{n}}$$

$$\mathcal{L} = \{ (1,-5,6) + t\langle 7,-17,-1\rangle \mid t \in \mathbb{R} \}$$

$x = 1 + 7t$
$y = -5 - 17t$
$z = 6 - t$

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INTERSECCIÓN DE PLANOS

- Se dice que dos planos son paralelos si sus normales son paralelas.

Teorema: Si π_1 y π_2 son planos paralelos, entonces $\pi_1 = \pi_2$ o bien $\pi_1 \cap \pi_2 = \emptyset$.

Si π_1 y π_2 NO son paralelos, entonces $\pi_1 \cap \pi_2$ es una recta.

$$\left.\begin{array}{l} - \pi_1 : \{ \overrightarrow{P_0P} \cdot \vec{n_1} = 0 \} \\ - \pi_2 : \{ \overrightarrow{P_0'P} \cdot \vec{n_2} = 0 \} \end{array}\right]$$

$$\boxed{\vec{n_1} \perp \vec{n_2}} = \vec{n_1} \cdot \vec{n_2} = 0$$

$$\boxed{\vec{n_1} \parallel \vec{n_2}} = \vec{n_1} = a\vec{n_2}$$

Dato: π_1 ($\vec{n_1}$, Ec 1) y π_2 ($\vec{n_2}$, Ec 2) $\Rightarrow$ Resolución de dos ecuaciones:

$$\begin{bmatrix} a_1 x + b_1 y + c_1 z + d_1 = 0 \\ a_2 x + b_2 y + c_2 z + d_2 = 0 \end{bmatrix}$$

EJERCICIO

- Hallar el ángulo entre los dos planos:

$\pi_1 : x - 2y + z = 0.$
$\pi_2 : 2x + 3y - 2z = 0$

$\vec{n_1} = \langle 1, -2, 1 \rangle$ $\qquad$ $\vec{n_2} = \langle 2, 3, -2 \rangle$

$$\theta = \arccos \left(\frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\| \|\vec{n_2}\|} \right)$$

$$\|\vec{n_1}\| = \sqrt{1+4+1} = \sqrt{6}$$
$$\|\vec{n_2}\| = \sqrt{4+9+4} = \sqrt{17}$$

$$\theta = \arccos \left(\frac{2 - 6 - 2}{\sqrt{6} \cdot \sqrt{17}} \right)$$

$$\theta = \arccos \frac{|-6|}{\sqrt{6 \cdot 17}} = \arccos \frac{6}{\sqrt{102}}$$

$$\boxed{\theta \approx 53.55^\circ}$$

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- Halle las ecuaciones paramétricas de su recta de intersección.

$$\Pi_1 \cap \Pi_2 = \left.\begin{aligned} x - 2y + z &= 0 \\ 2x + 3y - 2z &= 0 \end{aligned}\right\} \qquad \begin{aligned} -2x + 4y - 2z &= 0 \\ 2x + 3y - 2z &= 0 \\ \hline 7y - 4z &= 0 \end{aligned}$$

$$\boxed{y = \frac{4}{7}z}$$

$$\hookrightarrow \quad x - 2y + z = 0$$

$$x - 2\left(\tfrac{4}{7}\right) + z = 0$$

$$x - \tfrac{8}{7}z + \underbrace{z}_{7/7} = 0$$

$$\boxed{x = \tfrac{1}{7}z} \qquad \boxed{z = z}$$

∴ Si $t = \frac{z}{7}$, entonces

$$\boxed{\begin{aligned} x &= t \\ y &= 4t \\ z &= 7t \end{aligned}}$$

por conveniencia

ECUACIONES PARAMÉTRICAS DE LA RECTA

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{vmatrix} = \hat{i}\left[4-3\right] - \hat{j}\left[-2+2\right] + \hat{k}\left[3+4\right]$$

$$= \hat{i} + 4\hat{j} + 7\hat{k}$$

$$\boxed{= \langle 1, 4, 7 \rangle = \vec{v}}$$

$$\mathcal{L} = \{ P_0 + \vec{v}t \mid t \in \mathbb{R} \}$$

$$\mathcal{L} = \{ (0,0,0) + \langle 1,4,7 \rangle t \mid t \in \mathbb{R} \}$$

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- Encuéntrense los puntos de intersección de los dos planos.

$\Pi_1 = 4x + 3y + z = 0$. $\Pi_2 = x + y - z = 15$

$$\left.\begin{array}{r} 4x + 3y + \cancel{z} = 0 \\ x + y - \cancel{z} = 15 \end{array}\right\} \quad 5x + 4y = 15$$

$$x = \frac{15 - 4y}{5}$$

$$\boxed{x = 3 - \tfrac{4}{5}y}$$

$$4x + 3y + z = 0$$

$$4\left(3 - \tfrac{4}{5}y\right) + 3y + z = 0$$

$$12 - \tfrac{16}{5}y + 3y + z = 0$$

$$12 - \tfrac{1}{5}y + z = 0$$

$$\boxed{z = \tfrac{1}{5}y - 12}$$

$$\boxed{y = y}$$

- Tomamos $t = \frac{y}{5}$ entonces

$$x = 3 - 4t$$
$$y = 5t$$
$$z = t - 12 = -12 + t$$

$$\boxed{\mathscr{L}_1 = \{ (3, 0, -12) + \langle -4, 5, 1 \rangle t \mid t \in \mathbb{R} \}}$$